FINITE GENERATION OF THE COHOMOLOGY OF QUOTIENTS OF PBW ALGEBRAS

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Abstract. In this article we prove finite generation of the cohomology of quotients of a PBW algebra A by relating it to the cohomology of quotients of a quantum symmetric algebra S which is isomorphic to the associated graded algebra of A. The proof uses a spectral sequence argument and a finite generation lemma adapted from Friedlander and Suslin.

1. Introduction

The cohomology ring of a finite group is finitely generated, as proven by Evens [7], Golod [9] and Venkov [20]. The door to use geometric methods in the study of cohomology and modular representations of finite groups was opened due to this fundamental result. The cohomology ring of any finite group scheme (equivalently, finite dimensional cocommutative Hopf algebra) over a field of positive characteristic is finitely generated, as proven by Friedlander and Suslin [9], which is a generalization of the result of Venkov and Evens. In [11], Ginzburg and Kumar proved that cohomology of quantum groups at roots of unity is finitely generated. In [6], Etingof and Ostrik conjectured finite generation of cohomology in the context of finite tensor categories. The task of proving this conjecture was done by Mastnak, Pevtsova, Schauenburg and Witherspoon [15] for some classes of noncocommutative Hopf algebras over a field of characteristic 0 .

In [15], Mastnak, Pevtsova, Schauenburg and Witherspoon considered the Nichols algebra R. A finite filtration on R is used to define a spectral sequence to which they apply a finite generation lemma adapted from [8]. In order to do so, they define 2-cocycles on R that are identified with permanent cycles in the spectral sequence. Finally, they identify the permanent cycles belonging to the degree 2 cohomology of the associated graded algebra of R, with elements in the cohomology of S (where S is a quantum symmetric algebra subject to the relation $x_i^{N_i} = 0$ for all i) constructed in Section 4 of [15].

In this article, we generalize the work done by Mastnak, Pevtsova, Schauenburg and Witherspoon [15], by choosing our parameters that are not necessarily roots of unity and we allow non-nilpotent generators. Also we deal with PBW algebras in general, whereas in [15] the authors looked at those that arise from subalgebras of pointed Hopf algebras. Let k be a field, usually assumed to be algebraically closed and of characteristic 0. Let B be a PBW algebra over k generated by $x_1, \cdots, x_t, \cdots, x_n$ and $A = B/(x_1^{N_1}, \cdots, x_t^{N_t})$ where for each i, $1 \leq i \leq t$, N_i is an integer greater than 1 and $x_i^{N_i}$ is in the braided center. Our proof of finite generation of cohomology of the algebra A, is a two step procedure. First, we compute cohomology explicitly via a free S-resolution, where S is a quotient of a quantum symmetric

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algebra by the ideal generated by $x_1^{N_1}, \cdots, x_t^{N_t}$ where $1 \leq t \leq n$. Second, our algebra A has a filtration [4, Theorem 4.6.5] for which the associated graded algebra $(\mathrm{Gr}A)$ is S.

This work can potentially be applied to many algebras having PBW-like bases, possibly in combination with other techniques. For example, in Section 6 of [15], the authors used additional techniques to study the related algebras of somewhat different form. Apart from this, a few of the algebras of interest having such bases are the Frobenius-Lusztig kernels studied by Drupieski [5], pointed Hopf algebras studied by Helbig [12] and algebras studied by Liu [14].

Notation: $\mathrm{H}^r(A,k) = \mathrm{Ext}^r_A(k,k)$ and $\mathrm{H}^*(A,k) = \bigoplus_{r \geq 0} \mathrm{H}^r(A,k)$. The set $\mathbb{N}$ is assumed to contain 0. $E_r^{p,q}$ denotes the page r of the spectral sequence at position p,q and d_r is a map $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Main Theorem: The cohomology algebra $\mathrm{H}^*(A,k)$ is finitely generated.

We use the techniques of Mastnak, Pevtsova, Schauenberg and Witherspoon [15] to yield results in this general setting. However some difference do arise, notably we cannot apply [15, Lemma 2.5] as it is since our parameters are not necessarily roots of unity.

Organization: This article is organized as follows.

In Section 2 we define PBW algebras. In addition, we introduce a result from Evens [7] and a non-commutative version of a finite generation lemma adapted from Friedlander and Suslin [8].

Section 3 introduces a 2-cocycle on the algebra A. In Section 4, we prove that cohomology of the algebra A is finitely generated.

2. Definitions and Preliminary Results

2.1. **PBW Algebras.** In this subsection we recall some basic definitions including that of a PBW algebra.

Definition 2.1. An admissible ordering on $\mathbb{N}^n$ is a total ordering $<$ such that
1) if $\alpha < \beta$ and $\gamma \in \mathbb{N}^n$ then $\alpha + \gamma < \beta + \gamma$
2) $<$ is a well ordering.

This definition provides one-to-one correspondence between $\mathbb{N}^n$ and monomials in $k[x_1, \cdots, x_n]$. Some examples of ordering on n-tuples include:

Example 2.2. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$. The lexicographic order $<_{lex}$ on $\mathbb{N}^n$ is defined by letting $\beta <_{lex} \alpha$ if the first non zero entry of $\alpha - \beta \in \mathbb{Z}^n$ is positive.

For more examples of ordering on n-tuples we refer the reader to [4]. In light of this definition and example we define a PBW algebra.

Poincaré-Birkhoff-Witt Algebra: A PBW algebra R, over a field k, is a k-algebra together with elements $x_1, \cdots, x_n \in R$ and an admissible order on $\mathbb{N}^n$ for which there are scalars $q_{ij} \in k^*$ such that

1) $\{x_1^{\alpha_1}\cdots x_n^{\alpha_n} \mid (\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\}$ is a basis of R as a k-vector space. We call this basis the PBW basis.
2) $x_ix_j = q_{ij}x_jx_i + p_{ij}$ for $p_{ij}\in R$ with $exp(p_{ij}) < \varepsilon_i+\varepsilon_j$ $(1\le i<j\le n)$ where $\varepsilon_i = (0,\cdots,0,1_i,0,\cdots,0)\in\mathbb{N}^n$. (Notation "*exp*" is defined below).

Notation: By the basis condition of the definition, every $f\in R$ may be written uniquely as $f=\sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ (notation $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$) and $exp(f) = max\{\alpha\in\mathbb{N}^n \mid c_\alpha\neq 0\}$.

Let us now give some examples of PBW algebras.

Example 2.3. 1) The polynomial ring $R=k[x_1,x_2,\cdots,x_n]$ is a PBW algebra.
2) There are some quantum groups which are PBW algebras. For example:
a) The quantum plane $k_q[x,y] = k\langle x,y \mid yx = qxy\rangle$
b) $U_q(sl_3)^+ := k\langle x_1,x_2,x_3 \mid x_1x_2 = qx_2x_1,\ x_2x_3 = qx_3x_2,\ x_1x_3 = q^{-1}x_3x_1 + x_2\rangle$
3) Quantum Symmetric Algebra: Let k be a field. Let n be a positive integer and for each pair i,j of elements in $\{1,\cdots,n\}$, let q_{ij} be a nonzero scalar such that $q_{ii}=1$ and $q_{ji}=q_{ij}^{-1}$ for all i,j. Denote by $\mathbf{q}$ the corresponding tuple of scalars, $\mathbf{q} := (q_{ij})_{1\le i<j\le n}$. Let V be a vector space with basis $x_1,\cdots,x_n$, and let

$$S_{\mathbf{q}}(V) := k\langle x_1,\ldots,x_n \mid x_ix_j = q_{ij}x_jx_i \text{ for all } 1\le i<j\le n\rangle,$$

the *quantum symmetric algebra* (quantum polynomial ring) determined by $\mathbf{q}$.

The ω-filtration of a PBW algebra:
Let $\omega = (\omega_1,\cdots,\omega_n)\in\mathbb{N}^n$. For $0\neq f$ belonging to a PBW algebra R we define its ω-degree as

$$\deg_\omega(f) = \max\{|\alpha|_\omega \mid \alpha\in\mathcal{W}\}$$

where $|\alpha|_\omega = \alpha_1\omega_1+\cdots+\alpha_n\omega_n$, $f=\sum_{\alpha\in\mathbb{N}^n} c_\alpha x^\alpha$ and $\mathcal{W} = \{\alpha\in\mathbb{N}^n \mid c_\alpha\neq 0\}$. With these notations we define the ω-filtration of a PBW algebra as

$$F_s^\omega R = \{f\in R \mid |\alpha|_\omega \le s \text{ for all } \alpha\in\mathcal{W}\}$$

where s is any nonnegative integer (See [4]).

2.2. **Noetherian Modules.** Given a ring R, a decreasing filtration F^nR for $n\in\mathbb{N}$ is called compatible with the ring structure on R if $F^mR\cdot F^nR\subset F^{m+n}R$, for all $m,n\in\mathbb{N}$. The ring R with this filtration is then called a filtered ring (See [3]). Let $R=F^0R\supseteq F^1R\supseteq\cdots\supseteq F^sR\supseteq\cdots$ be a graded filtered ring. Note that by definition, the grading on R is compatible with its ring structure in the usual way that is $R=\bigoplus_{n\in\mathbb{N}} R^n$, and $R^nR^m\subset R^{n+m}$. Then we may form the doubly graded ring

$$E_0(R) = \sum_i F^iR/F^{i+1}R.$$

Similarly, we may form the doubly graded module $E_0(N)$ over $E_0(R)$ if N is a graded filtered module over R, (with the module structure consistent with the ring structure in the usual way that is $N=\bigoplus_{i\in\mathbb{N}} N^i$, and $R^iN^j\subset N^{i+j}$).

For the current purposes it is sufficient to consider filtrations such that $F^iR^n = 0$ for i sufficiently large where n denotes the grading on R. Similarly, $F^iN^j = 0$ for i sufficiently large.

Now we define a couple of terms and recall the following proposition of Evens [7].

Definition 2.4. 1) A submodule S of an R-module N is said to be homogeneous if it is generated by homogeneous elements (i.e. elements from homogeneous summands N^i).

2) An R-submodule N of a graded R-module M is called a graded R-submodule of M if we have $N = \bigoplus_s (N \cap M^s)$.

3) If $\{F^s M\}$ is a filtration of the R-module M, and N is a submodule of M, then we have a filtration induced on N, given by $F^s N = N \cap F^s M$.

Proposition 2.5. *Let R be a graded filtered ring i.e.*

$$R = F^0 R \supseteq F^1 R \supseteq \cdots \supseteq F^s R \supseteq \cdots$$

and N a graded filtered R module i.e. suppose

$$N = F^0 N \supseteq F^1 N \supseteq \cdots \supseteq F^s N \supseteq \cdots$$

over R. If $E_0(N)$ is (left) Noetherian over $E_0(R)$, then N is Noetherian over R.

Proof. See [7, Section 2, Proposition 2.1] and [19, Chapter 2]. □

A finite generation lemma. In Section 4, we will need the following general lemma which is a non-commutative version of [15, Lemma 2.5] and is originally adapted from [8, Lemma 1.6]. Recall that an element $x \in E_r^{p,q}$ is called a *permanent cycle* if $d_i(x) = 0$ for all $i \geq r$.

Lemma 2.6. *a) Let $E_1^{p,q} \Rightarrow E_\infty^{p+q}$ be a multiplicative spectral sequence of bigraded k-algebras concentrated in the half plane $p + q \geq 0$ and let $C^{*,*}$ be a bigraded k-algebra. For each fixed q, assume that $C^{p,q} = 0$ for p sufficiently large. Assume that there exists a bigraded map of algebras $\phi : C^{*,*} \to E_1^{*,*}$ such that*
1) ϕ makes $E_1^{*,*}$ into a left Noetherian $C^{*,*}$-module, and
2) the image of $C^{*,*}$ in $E_1^{*,*}$ consists of permanent cycles.
Then E_∞^* is a left Noetherian module over Tot($C^{*,*}$).
b) Let $\widetilde{E}_1^{p,q} \Rightarrow \widetilde{E}_\infty^{p+q}$ be a spectral sequence that is a bigraded module over the spectral sequence $E^{*,*}$. Assume that $\widetilde{E}_1^{*,*}$ is a left Noetherian module over $C^{*,*}$ where $C^{*,*}$ acts on $\widetilde{E}_1^{*,*}$ via the map ϕ. Then $\widetilde{E}_\infty^*$ is a finitely generated E_∞^*-module.

Proof. Let $\Lambda_r^{*,*} \subset E_r^{*,*}$ be the bigraded subalgebra of permanent cycles in $E_r^{*,*}$.
We claim first that $d_r(E_r^{*,*}) \subset \Lambda_r^{*,*}$. In order to see this note that $d_r(E_r^{*,*}) = \mathrm{im}(d_r)$. Therefore, $d_r(E_r^{*,*}) \subset \mathrm{Ker}\ d_{r+1}$. Hence, $d_{r+1}(d_r(E_r^{*,*})) = 0$. Similarly, $d_{r+2}(d_r(E_r^{*,*})) = 0$ and so on. Thus, we have $d_i(d_r(E_r^{*,*})) = 0$ for all $i \geq r$. Hence, $d_r(E_r^{*,*}) \subset \Lambda_r^{*,*}$.

Next we claim that for all $\lambda \in \Lambda_r^{*,*}$ and $\mu \in E_r^{*,*}$, $\lambda \cdot d_r(\mu) \in d_r(E_r^{*,*})$ that is, $d_r(E_r^{*,*})$ is a left ideal of $\Lambda_r^{*,*}$. Consider

$$\begin{aligned} d_r(\lambda \cdot \mu) &= d_r(\lambda)\mu + (-1)^{p+q}\lambda \cdot d_r(\mu) \quad \text{where } \lambda \in \Lambda^{p,q} \\ &= 0 + (-1)^{p+q}\lambda \cdot d_r(\mu) \end{aligned}$$

So $\lambda \cdot d_r(\mu) \in d_r(E_r^{*,*})$. Thus $d_r(E_r^{*,*})$ is a left ideal of $\Lambda_r^{*,*}$.

Now the image of $C^{*,*}$ is contained in each page of the spectral sequence and by assumption it consists of permanent cycles. Hence, we can similarly conclude as above that $d_r(E_r^{*,*})$ is a $C^{*,*}$-submodule.

A similar computation as above shows that $\Lambda_1^{*,*}$ is a $C^{*,*}$-submodule of $E_1^{*,*}$. To see this let $a \in C^{p,q}$; then $\phi(a) \in E_1^{*,*}$ and $\lambda_1 \in \Lambda_1^{*,*}$. Consider

$$\begin{aligned} d_i(\phi(a)\lambda_1) &= d_i(\phi(a))\lambda_1 + (-1)^{p+q}\phi(a)d_i(\lambda_1) \quad \text{where } i \geq 1 \\ &= \; 0 + 0 = 0 \end{aligned}$$

So $\phi(a)\lambda_1 \in \Lambda_1^{*,*}$. Thus $\Lambda_1^{*,*}$ is a $C^{*,*}$-submodule.

By induction, $\Lambda_{r+1}^{*,*} = \Lambda_r^{*,*}/d_r(E_r^{*,*})$ is a $C^{*,*}$-module for any $r \geq 1$ because $d_r(E_r^{*,*}) \subset \Lambda_r^{*,*}$ and by the induction hypothesis $\Lambda_r^{*,*}$ is a $C^{*,*}$-module. Therefore, $\Lambda_r^{*,*}/d_r(E_r^{*,*})$ is a $C^{*,*}$-module that is, $\Lambda_{r+1}^{*,*}$ is a $C^{*,*}$-module.

We get a sequence of surjective maps of $C^{*,*}$-modules:

$$\Lambda_1^{*,*} \twoheadrightarrow \Lambda_2^{*,*} \twoheadrightarrow \cdots \twoheadrightarrow \Lambda_r^{*,*} \twoheadrightarrow \Lambda_{r+1}^{*,*} \twoheadrightarrow \cdots \tag{2.1}$$

Since $\Lambda_1^{*,*}$ is a $C^{*,*}$-submodule of $E_1^{*,*}$, it is Noetherian as a $C^{*,*}$-module. Therefore, the kernels of the maps $\Lambda_1^{*,*} \twoheadrightarrow \Lambda_r^{*,*}$ are Noetherian for all $r \geq 1$. These kernels form an increasing chain of submodules of $\Lambda_1^{*,*}$; hence, by the Noetherian property, they stabilize after finitely many steps; that is, $\Lambda_r^{*,*} = \Lambda_{r+1}^{*,*} = \cdots$ for some r. We conclude that $\Lambda_r^{*,*} = E_\infty^{*,*}$. Therefore $E_\infty^{*,*}$ is a Noetherian $C^{*,*}$-module. Also, both $E_\infty^{*,*}$ and $C^{*,*}$ are filtered algebras and the filtration for each n is given by:

$$E_\infty^n = \bigoplus_{p+q=n} E_\infty^{p,q} \supseteq \bigoplus_{\substack{p+q=n \\ p\geq 1}} E_\infty^{p,q} \supseteq \bigoplus_{\substack{p+q=n \\ p\geq 2}} E_\infty^{p,q} \supseteq \cdots$$

and $E_\infty^{*,*}$ is the associated graded algebra. Similarly, for each n:

$$C^n = \bigoplus_{p+q=n} C^{p,q} \supseteq \bigoplus_{\substack{p+q=n \\ p\geq 1}} C^{p,q} \bigoplus_{\substack{p+q=n \\ p\geq 2}} C^{p,q} \supseteq \cdots$$

and $C^{*,*}$ is the associated graded algebra.

For p sufficiently large, $C^{p,q} = 0$. Hence, by proposition 2.5, E_∞^* is a Noetherian module over $\mathrm{Tot}(C^{*,*})$.

(b) Similarly, we can show that $\widetilde{E}_\infty^{*,*}$ is Noetherian over $C^{*,*}$. Again, by applying Proposition 2.5, we can conclude that $\widetilde{E}_\infty^*$ is Noetherian and hence finitely generated over $\mathrm{Tot}(C^{*,*})$. Therefore, by part (a) $\widetilde{E}_\infty^*$ is a Noetherian module over E_∞^*. Hence, $\widetilde{E}_\infty^*$ is finitely generated over E_∞^*.

□

3. Some Cocycles on The Algebra

For this section, we will use the same terminology as used by Mastnak and Witherspoon in Section 6 of [16], with some additional information.

Let B be a PBW algebra over k as defined in Section 2 and $A = B/(x_1^{N_1}, \cdots, x_t^{N_t})$. As a vector space B has a basis $\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid i_1, \cdots, i_n \in \mathbb{N}\}$.

We want to show that the above set is indeed a basis for A with some restriction on $i_j, 1 \leq j \leq t$. To prove it is a basis we need the assumption that $x_i^{N_i}$ is in the braided center (defined below) of B for all $i,\ 1 \leq i \leq t$.

Let $b \in B$. Then $b = \sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ is a finite sum where $I = (i_1, i_2, \cdots, i_n)$ and a_I is a scalar. Therefore,

$$\begin{aligned} b + (x_1^{N_1}, \cdots, x_t^{N_t}) &= \sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + (x_1^{N_1}, \cdots, x_t^{N_t}) \\ &= \sum_{\substack{I \\ 0 \leq i_j < N_j \\ 1 \leq j \leq t}} (a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + (x_1^{N_1}, \cdots, x_t^{N_t})) \end{aligned}$$

This proves that $\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid 0 \leq i_1 < N_1, \cdots, 0 \leq i_t < N_t,\ i_{t+1}, \cdots, i_n \in \mathbb{N}\}$ is a spanning set for A.

Define,

$$[x_1^{i_1} \cdots x_n^{i_n}, x_1^{j_1} \cdots x_n^{j_n}]_c = x_1^{i_1} \cdots x_n^{i_n} x_1^{j_1} \cdots x_n^{j_n} - (\prod_{k<l} q_{lk}^{-(j_l i_k - j_k i_l)}) x_1^{j_1} \cdots x_n^{j_n} x_1^{i_1} \cdots x_n^{i_n}.$$

Definition 3.1. An element of the form $x_1^{i_1} \cdots x_n^{i_n}$ is said to be in the braided center of B, if

$$[x_1^{i_1} \cdots x_n^{i_n}, x_1^{j_1} \cdots x_n^{j_n}]_c = 0, \text{ for all } x_1^{j_1} \cdots x_n^{j_n} \in B. \tag{3.1}$$

Assume that $x_i^{N_i}$ is in the braided center of B for all $i,\ 1 \leq i \leq t$. We will also need this assumption for a later part of this section.

To show that the set is linearly independent we need to prove that $\sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ belonging to $(x_1^{N_1}, \cdots, x_t^{N_t})$ implies all $a_I = 0$.

Consider

$$\sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = \sum_{J,i} T_J x_i^{N_i} W_J$$

where $T_J, W_J \in B$. Since $x_i^{N_i}$ is in the braided center we have

$$\sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} = \sum_{J,i} x_i^{N_i} U_J$$

where $U_J \in B$. Observe that in each expression on the right hand side there is at least one i for which the power of x_i is at least N_i. Thus by comparing the coefficients we get $a_I = 0$. Hence, $\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid 0 \leq i_1 < N_1, \cdots, 0 \leq i_t < N_t,\ i_{t+1}, \cdots, i_n \in \mathbb{N}\}$ is a basis for A.

Next we want to define 2-cocycles ζ_i on A. These 2-cocycles represent the elements of $\mathrm{H}^2(A, k)$. We make use of the reduced bar resolution of k,

$$\cdots \longrightarrow B \otimes (B^+)^{\otimes 2} \xrightarrow{\delta_2} B \otimes B^+ \xrightarrow{\delta_1} B \xrightarrow{\varepsilon} k \longrightarrow 0.$$

where B is an augmented algebra with augmention map $\varepsilon : B \to k$, $B^+ = \mathrm{Ker}\ \varepsilon$ is the augmentation ideal and $\delta_i(b_0 \otimes b_1 \otimes \cdots \otimes b_i) = \sum_{j=0}^{i-1} (-1)^j b_0 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_i$. For each $i, 1 \leq i \leq t$ define $\tilde{\zeta}_i : B^+ \otimes B^+ \to k$ by

$$\tilde{\zeta}_i(r \otimes s) = \gamma_{(0,\cdots,0,N_i,0,\cdots,0)}$$

where N_i is in the i^{th} position and $rs = \sum_a \gamma_a x^a \in B$. We need to check that $\tilde{\zeta}_i(r \otimes s)$ is associative, that is to show that $\tilde{\zeta}_i(rr_1 \otimes s) = \tilde{\zeta}_i(r \otimes r_1 s)$ for all $r, r_1, s \in B^+$. But this is true by definition and thus $\tilde{\zeta}_i$ may be trivially extended to a 2-cocycle on B. Let us see how it is done. We will denote the 2-cocycle on B by $\tilde{\zeta}_i$ and define as $\tilde{\zeta}_i(b_1 \otimes b_2) = \tilde{\zeta}_i \mid_{B^+ \otimes B^+} (b_1 \otimes b_2)$ for $b_1, b_2 \in B^+$. Indeed $\tilde{\zeta}_i$ is a coboundary on B that is $\tilde{\zeta}_i = -\delta^* h_i$ where $h_i(r)$ is the coefficient of $x_i^{N_i}$ in $r \in B^+$ written as a linear combination of PBW basis elements. To see this note that $h_i : B \otimes B^+ \to k$ is a 1-cochain, $\mathrm{Hom}_B(B \otimes B^+, k) \cong \mathrm{Hom}_k(B^+, k)$ and $\delta^* h_i \in \mathrm{Hom}_B(B \otimes B^+ \otimes B^+, k)$.

To define a 2-cocycle ζ_i on A, we next show that $\tilde{\zeta}_i$ factors through the quotient map $\pi : B \to A$ and that ζ_i is not a coboundary on A. We must show that $\tilde{\zeta}_i(r, s) = 0$ whenever either r or $s \in \mathrm{Ker}\ \pi$. Consider the following diagram

$$\begin{tikzcd} B^+ \otimes B^+ \arrow[r, "\tilde{\zeta}_i"] \arrow[d, "\pi\otimes\pi"'] & k \\ A \otimes A \arrow[ur, dotted, "\zeta_i"'] & \end{tikzcd}$$

Suppose $x^a \in \mathrm{Ker}\ \pi$ then $a_j \geq N_j$ for some j with $1 \leq j \leq t$. As per the assumption that $x_i^{N_i}$ is in the braided center, we can write $x^a = \vartheta x_j^{N_j} x^b$ where ϑ is a non-zero scalar and b is arbitrary. Therefore, $\tilde{\zeta}_i(x^a \otimes x^c) = \vartheta \tilde{\zeta}_i(x_j^{N_j} x^b \otimes x^c)$ and this is the coefficient of $x_i^{N_i}$ in the product $\vartheta x_j^{N_j} x^b x^c$. If $j = i$, then since $x^c = x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \in B^+$ the above product cannot have non-zero coefficient for $x_i^{N_i}$. The same is true, if $j \neq i$ since $x_j^{N_j}$ is a factor of $x^a x^c$. If $x^c \in \mathrm{Ker}\ \pi$ a similar argument will work.

Thus, we have $\tilde{\zeta}_i(x^a \otimes x^c) = 0$ that is, $\tilde{\zeta}_i$ factors through the quotient map $\pi : B \to A$. Therefore, we may define $\zeta_i : A^+ \otimes A^+ \to k$ by

$$\zeta_i(r \otimes s) = \tilde{\zeta}_i(\tilde{r} \otimes \tilde{s})$$

where $\tilde{r}, \tilde{s}$ are defined via a section of π. (Choose the section ϕ of the quotient map $\pi : B \to A$ such that $\phi(r) = \tilde{r}$ where $\tilde{r}$ is the unique element that is, a linear combination of the PBW basis elements of B with $i_l < N_l$ for all $l = 1, \cdots, n$).

This is well defined since $\tilde{\zeta}_i$ is well defined. We still need to verify that ζ_i is associative on A^+. Let $r, s, u \in A^+$ and since π is algebra homomorphism, $\tilde{r}\tilde{s} = \widetilde{rs} + y$ and $\tilde{s}\tilde{u} = \widetilde{su} + z$ for some $y, z \in \mathrm{Ker}\ \pi$. Observe that $\mathrm{Ker}\ \pi \otimes B + B \otimes \mathrm{Ker}\ \pi \subset \mathrm{Ker}\ \tilde{\zeta}_i$.
Therefore, we have

$$\begin{aligned} \zeta_i(rs \otimes u) &= \tilde{\zeta}_i(\widetilde{rs} \otimes \tilde{u}) \\ &= \tilde{\zeta}_i((\tilde{r}\tilde{s} - y) \otimes \tilde{u}) \\ &= \tilde{\zeta}_i(\tilde{r}\tilde{s} \otimes \tilde{u}) \\ &= \tilde{\zeta}_i(\tilde{r} \otimes \tilde{s}\tilde{u}) \quad (\tilde{\zeta}_i \text{ associative }) \\ &= \tilde{\zeta}_i(\tilde{r} \otimes \widetilde{su}) \\ &= \zeta_i(r \otimes su) \end{aligned}$$

This shows that ζ_i is associative on A^+. Hence, ζ_i is 2-cocycle on A.

4. Finite Generation

In this section we prove our main theorem. We follow the same terminology as used in Section 5 of [15] with some additional information.

Let B be a PBW algebra as defined in Section 2 and $A = B/(x_1^{N_1}, \cdots, x_t^{N_t})$. Recall the assumption from Section 3 that $x_i^{N_i}$ is in the braided center. Hence, a filtration on B induces a filtration on A [4, Theorem 4.6.5] for which $S = GrA$, given by generators and relations of type

$$S = k\langle x_1, \cdots, x_t, \cdots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i < j \text{ and } x_i^{N_i} = 0 \text{ for } 1 \leq i \leq t\rangle,$$

where $1 < N_i \in \mathbb{Z}$, and $q_{ij} \in k^*$ for $1 \leq i < j \leq n$ with $q_{ji} = q_{ij}^{-1}$ for $i < j$ and $q_{ii} = 1$. Thus $\mathrm{H}^*(S, k)$ is given by the following theorem from [19, Theorem 3.1] (cf.[2, Theorem 5.3]):

Theorem 4.1. *Let S be the k-algebra generated by $x_1, \cdots, x_n$, subject to relations $x_i x_j = q_{ij} x_j x_i$ for all $i < j$, $x_i^{N_i} = 0$ for $1 \leq i \leq t$. Then $\mathrm{H}^*(S, k)$ is generated by ξ_i $(i = 1, \cdots, t)$ and η_i $(i = 1, \cdots, n)$ where deg $\xi_i = 2$ and deg $\eta_i = 1$, subject to the relations*

$$\xi_i \xi_j = q_{ji}^{N_i N_j} \xi_j \xi_i, \ \ \eta_i \xi_j = q_{ji}^{N_j} \xi_j \eta_i, \quad \text{and} \quad \eta_i \eta_j = -q_{ji} \eta_j \eta_i$$

and

$$\eta_i^2 = 0 \text{ if } N_i \neq 2 \text{ and } \eta_i^2 \text{ is a nonzero scalar multiple of } \xi_i \text{ if } N_i = 2.$$

Now our algebra A is an augmented algebra over the field k, with augmentation $\varepsilon : A \to k$. Since A is filtered it induces an increasing filtration $F_0 P_\bullet \subset F_1 P_\bullet \subset \cdots \subset F_n P_\bullet \subset \cdots$ on the reduced bar (free A) resolution of k,

$$P_\bullet : \cdots \xrightarrow{\partial_3} A \otimes (A^+)^{\otimes 2} \xrightarrow{\partial_2} A \otimes A^+ \xrightarrow{\partial_1} A \xrightarrow{\varepsilon} k \to 0$$

where $A^+ = \mathrm{Ker}\ \varepsilon$, $\partial_n(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n$ and the filtration is given in each degree n by

$$F_p(A \otimes (A^+)^{\otimes n}) = \sum_{i_0 + \cdots + i_n = p} F_{i_0} A \otimes F_{i_1}(A^+) \otimes \cdots \otimes F_{i_n}(A^+).$$

Then the reduced bar complex of GrA is precisely $GrP_\bullet$, where

$$(GrP_n)_p := F_p P_n / F_{p-1} P_n.$$

Now let $\mathcal{C}^\bullet(A) := \mathrm{Hom}_A(P_\bullet, k)$. Note that $\mathcal{C}^n(A) = \mathrm{Hom}_A(P_n, k) = \mathrm{Hom}_A(A \otimes (A^+)^{\otimes n}, k)$ is a filtered vector space where

$$F^p \mathcal{C}^n(A) = \{f : P_n \to k \mid f\,|_{F_{p-1}P_n} = 0\}$$

This filtration is compatible with the coboundary map on $\mathcal{C}^\bullet(A)$. Hence, $\mathcal{C}^\bullet(A)$ is a filtered cochain complex: $\mathcal{C} = F^0 \mathcal{C}^\bullet \supset F^1 \mathcal{C}^\bullet \supset \cdots$. Now our algebra A satisfies $F_p A = 0$ if $p < 0$, $1 \in F_0 A$ and $A = \bigcup_p F_p A$. Thus, there is a convergent May spectral sequence associated to the filtration of a cochain complex (see [17, Theorem 3] and [18, Theorem 12.5]):

$$E_1^{p,q} = \mathrm{H}^{p+q}((GrA)_p, k) \Longrightarrow \mathrm{H}^{p+q}(A, k). \tag{4.1}$$

Note: For special cases refer to [21, Theorem 5.5.1].

From Section 3 we know that

$$\zeta_i(x^a \otimes x^b) = \gamma_i \tag{4.2}$$

where γ_i is the coefficient of $x_i^{N_i}$ in the product $x^a x^b$, and x^a, x^b range over all pairs of PBW basis elements. Recall that any PBW basis element is written as $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. By [4, Theorem 4.6.5] we know that there exists a filtration and thus, there is a total ordering which we denote by p_i (a positive integer), which is a number associated to ζ_i. Now, observe that ζ_i is in degree $(p_i, 2-p_i)$. We wanted to relate these functions ζ_i to the elements of the E_1 page of the spectral sequence (4.1). We have $\zeta_i \mid_{F_{p_i-1}(A\otimes A)}= 0$ but $\zeta_i \mid_{F_{p_i}(A\otimes A)}\neq 0$ by (4.2). Thus, we conclude by the definition of ζ_i from Section 3 that $\zeta_i \in F^{p_i}\mathcal{C}^2$ but $\zeta_i \notin F^{p_i+1}\mathcal{C}^2$. The filtration on $\mathcal{C}^\bullet$ induces a filtration on $\mathrm{H}^*(\mathcal{C}^\bullet)$, that is to say $F^p\mathrm{H}^n(\mathcal{C}^\bullet) := \mathrm{im}\{\mathrm{H}^n(F^p\mathcal{C}^\bullet) \to \mathrm{H}^n(\mathcal{C}^\bullet)\}$ with $F^0\mathrm{H}^n(\mathcal{C}^\bullet) = \mathrm{H}^n(\mathcal{C}^\bullet)$. By denoting the corresponding cocycle in $F^{p_i}\mathrm{H}^2(A,k)$ by the same letter we further conclude that $\zeta_i \in \mathrm{im}\{\mathrm{H}^2(F^{p_i}\mathcal{C}^\bullet) \to \mathrm{H}^2(\mathcal{C}^\bullet)\} = F^{p_i}\mathrm{H}^2(A,k)$, but $\zeta_i \notin \mathrm{im}\{\mathrm{H}^2(F^{p_i+1}\mathcal{C}^\bullet) \to \mathrm{H}^2(\mathcal{C}^\bullet)\} = F^{p_i+1}\mathrm{H}^2(A,k)$. Hence, we can identify ζ_i with corresponding nontrivial homogeneous element in the associated graded complex:

$$\tilde{\zeta}_i \in F^{p_i}\mathrm{H}^2(A,k)/F^{p_i+1}\mathrm{H}^2(A,k) \simeq E_\infty^{p_i,2-p_i}.$$

We refer to [17] for the isomorphism.

Since $\zeta_i \in F^{p_i}\mathcal{C}^2$ but $\zeta_i \notin F^{p_i+1}\mathcal{C}^2$, it induces an element $\bar{\zeta}_i \in E_0^{p_i,2-p_i} = F^{p_i}\mathcal{C}^2/F^{p_i+1}\mathcal{C}^2$ which will be in the kernels of all the differentials of the spectral sequence since it is induced by an actual cocycle in $\mathcal{C}^\bullet$. Hence, the image of $\bar{\zeta}_i$ will be in the E_∞-page. Now the non-zero element $\tilde{\zeta}_i$ is also induced by the same cocycle as $\bar{\zeta}_i$ in $\mathcal{C}^\bullet$. Hence we may identify these cocycles. This leads to the conclusion that $\tilde{\zeta}_i \in E_0^{p_i,2-p_i}$, and, correspondingly, its image in $E_1^{p_i,2-p_i} \hookrightarrow \mathrm{H}^2(GrA,k)$ which we denote by the same symbol, is a permanent cycle.

Note that via the formula (4.2) we can obtain similar cocycles $\hat{\zeta}_i$ for $S = GrA$. Comparing the values of $\bar{\zeta}_i$ and $\hat{\zeta}_i$ on basis elements $x^a \otimes x^b$ of $GrA \otimes GrA$ leads us to the conclusion that they are the same function. Hence $\hat{\zeta}_i \in E_1^{p_i,2-p_i}$ are permanent cycles.

We will identify these elements $\hat{\zeta}_i \in \mathrm{H}^2(GrA,k)$ with the cohomology classes $\xi_i \in \mathrm{H}^*(S,k)$ of Theorem 4.1 via the following theorem.

Theorem 4.2. *For each i $(1 \le i \le n)$, the cohomology classes ξ_i and $\hat{\zeta}_i$ coincide as elements of $\mathrm{H}^2(GrA,k)$.*

Proof. In Section 3 of [19] the chain complex $K_\bullet$ which is a projective resolution of the trivial GrA-module k is defined as

$$K_m = \oplus_{a_1+\cdots+a_n=m} S\Phi(a_1,\cdots,a_n),$$

where for each n-tuple $(a_1,\cdots,a_n)$ of non-negative integers with $a_i = 0$ or 1 for each i, $t+1 \le i \le n$, $\Phi(a_1,\cdots,a_n)$ is a free generator in degree $a_1+\cdots+a_n$.
The differential of this complex is defined as:

For each $i,\ 1 \le i \le t$, let $\sigma_i, \tau_i : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma_i(a) = \begin{cases} 1, \text{ if } a \text{ is odd} \\ N_i - 1, \text{ if } a \text{ is even,} \end{cases}$$

and $\tau_i(a) = \sum_{j=1}^{a} \sigma_i(j)$ for $a \ge 1, \tau_i(0) = 0$. For each $i,\ t+1 \le i \le n$ we define $\sigma_i(a) = 1$ and $\tau_i(a) = a$. Then

$$d_i(\Phi(a_1, \cdots, a_n)) = \begin{cases} \prod_{i<l} (-1)^{a_l} q_{li}^{\sigma_i(a_i)\tau_l(a_l)} x_i^{\sigma_i(a_i)} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n), & \text{if } a_i > 0 \\ 0, & \text{if } a_i = 0 \end{cases}$$

Elements $\eta_i \in \mathrm{H}^1(GrA, k)$ and $\xi_i \in \mathrm{H}^2(GrA, k)$ is defined via the complex $K_\bullet$ as

$$\xi_i(\Phi(a_1, \cdots, a_n)) = \prod_{l<i} q_{il}^{N_i \tau_l(a_l)} \Phi(a_1, \cdots, a_i - 2, \cdots, a_n), \quad \text{if } 1 \le i \le t$$

$$\eta_i(\Phi(a_1, \cdots, a_n)) = \prod_{i<l} q_{li}^{(\sigma_i(a_i)-1)\tau_l(a_l)} \prod_{l<i} (-1)^{a_l} q_{il}^{\tau_l(a_l)} x_i^{\sigma_i(a_i)-1} \Phi(a_1, \cdots, a_i - 1, \cdots, a_n).$$

Our aim is to identify ξ_i with the elements of the chain complex $\mathcal{C}^\bullet$ defined above. For this we consider the following diagram and define the maps F_1, F_2 making it commutative, where $S = GrA$:

$$\begin{tikzcd} \cdots \arrow[r] & K_2 \arrow[r, "d"] \arrow[d, "F_2"] & K_1 \arrow[r, "d"] \arrow[d, "F_1"] & K_0 \arrow[r, "\varepsilon"] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & S \otimes (S^+)^{\otimes 2} \arrow[r, "\partial_2"] & S \otimes (S^+) \arrow[r, "\partial_1"] & S \arrow[r, "\varepsilon"] & k \arrow[r] & 0 \end{tikzcd}$$

where the map $d = d_1 + d_2 + \cdots + d_n$ and $\partial_i(s_0 \otimes s_1 \otimes \cdots \otimes s_i) = \sum_{j=0}^{i-1} (-1)^j s_0 \otimes \cdots \otimes s_j s_{j+1} \otimes \cdots \otimes s_i$ is defined in Section 3. Let $\Phi(\cdots 1_i \cdots)$ where 1 is in the ith position and 0 in all other positions denote the basis element of K_1, $\Phi(\cdots 1_i \cdots 1_j \cdots)$ (respectively $\Phi(\cdots 2_i \cdots)$ for $i \le t$) where 1 is in the ith and jth positions ($i \ne j$), and 0 in all other positions (respectively a 2 in the ith position and 0 in all other positions) denote the basis element of K_2. Let

$$\begin{aligned} F_1(\Phi(\cdots 1_i \cdots)) &= 1 \otimes x_i, \\ F_2(\Phi(\cdots 2_i \cdots)) &= \sum_{a_i=0}^{N_i-2} x_i^{a_i} \otimes x_i \otimes x_i^{N_i - a_i - 1}, \\ F_2(\Phi(\cdots 1_i \cdots 1_j \cdots)) &= 1 \otimes x_j \otimes x_i - q_{ji} \otimes x_i \otimes x_j \end{aligned}$$

We want to provide a chain map $F_\bullet : K_\bullet \to S \otimes (S^+)^{\otimes \bullet}$ by extending F_1, F_2 to maps $F_i : K_i \to S \otimes (S^+)^{\otimes i}, i \ge 1$. This can be done by showing that the two nontrivial squares in the above diagram commute.

Consider

$$\begin{aligned} d(\Phi(\cdots 1_i \cdots)) &= (d_1 + \cdots + d_i + \cdots + d_n)(\Phi(\cdots 1_i \cdots)) \\ &= x_i \Phi(\cdots 0_i \cdots) \\ &= x_i \end{aligned}$$

$$\begin{aligned} \partial_1 \circ F_1(\Phi(\cdots 1_i \cdots)) &= \partial_i(1 \otimes x_i) \\ &= 1 \cdot x_i \\ &= x_i \end{aligned}$$

Thus, we have $d = \partial_1 \circ F_1$. Similarly, we can check that $F_1 \circ d = \partial_2 \circ F_2$.

Hence, two nontrivial squares in the above diagram commute. So by the Comparison Theorem [13] there exists a chain map $F_\bullet : K_\bullet \to S \otimes (S^+)^{\otimes \bullet}$ that induces an isomorphism on cohomology.

We now verify that the maps F_1, F_2 give the desired identifications. Here we use the definition in (4.2) to represent the function ξ_i on the reduced bar complex, $\xi_i(1 \otimes x^a \otimes x^b) := \xi_i(x^a \otimes x^b)$. Then

$$\begin{aligned} F_2^*(\xi_i)(\Phi(\cdots 2_i \cdots)) &= \xi_i(F_2(\Phi(\cdots 2_i \cdots))) \\ &= \xi_i(\sum_{a_i=0}^{N_i-2} x_i^{a_i} \otimes x_i \otimes x_i^{N_i-a_i-1}) \\ &= \sum_{a_i=0}^{N_i-2} \varepsilon(x_i^{a_i}) \xi_i(1 \otimes x_i \otimes x_i^{N_i-a_i-1}) \\ &= \xi_i(x_i \otimes x_i^{N_i-1}) \\ &= 1 \end{aligned}$$

Similarly, we can check that $F_2^*(\xi_i)(\Phi(\cdots 1_i \cdots 1_j \cdots)) = 0$ for all i, j and $F_2^*(\xi_i)(\Phi(\cdots 2_j \cdots)) = 0$ for all $j \neq i$. Therefore, $F_2^*(\xi_i)$ is the dual function to $\Phi(\cdots 2_i \cdots)$ which is precisely ξ_i.

□

In the same manner, we identify the elements η_i defined above with functions at the chain level in cohomology. For that define

$$\eta_i(x^a) = \begin{cases} 1, & \text{if } x^a = x_i \\ 0, & \text{otherwise} \end{cases}$$

The functions η_i represent a basis of $\mathrm{H}^1(S, k) \simeq \mathrm{Hom}_k(S^+/(S^+)^2, k)$. Consider,

$$\begin{aligned} F_1^*(\eta_i)(\Phi(\cdots 1_j \cdots)) &= \eta_i(F_1(\Phi(\cdots 1_j \cdots))) \\ &= \eta_i(1 \otimes x_j) \\ &= \eta_i(x_j) \\ &= \begin{cases} 1, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases} \\ &= \delta_{ij} \end{aligned}$$

Thus $F_1^*(\eta_i)$ is the dual function to $\Phi(\cdots 1_i \cdots)$. Therefore η_i and $\hat{\eta_i}$ coincide as elements of $\mathrm{H}^1(S,k)$ where $\hat{\eta_i}$ is a 1-cocycle of A.

Theorem 4.3. *The cohomology algebra* $\mathrm{H}^*(A,k)$ *is finitely generated.*

Proof. Let $E_1^{*,*} \Longrightarrow \mathrm{H}^*(A,k)$ be the May spectral sequence and $D^{*,*}$ be the bigraded subalgebra of $E_1^{*,*}$ generated by the elements ξ_i. So by the above discussion $D^{*,*}$ consists of permanent cycles and ξ_i is in bidegree $(p_i, 2-p_i)$. Moreover, $D^{*,*}$ is Noetherian since it is a quantum polynomial algebra in ξ_i [10]. By Theorem 4.1 the algebra $E_1^{*,*}$ is generated by ξ_i and η_i where the generators η_i are nilpotent. Since $D^{*,*}$ is a subalgebra of $E_1^{*,*}$, we get an inclusion map $f : D^{*,*} \to E_1^{*,*}$ making $E_1^{*,*}$ a module over $D^{*,*}$. Hence, $E_1^{*,*}$ is a finitely generated module over $D^{*,*}$ and is generated by $\eta_1, \cdots, \eta_n$. Therefore, by Lemma 2.6, E_∞^* is a Noetherian $\mathrm{Tot}(D^{*,*})$-module. But $E_\infty^* \cong Gr\ \mathrm{H}^*(A,k)$ [17]. Thus, $Gr\ \mathrm{H}^*(A,k)$ is a Noetherian $\mathrm{Tot}(D^{*,*})$-module and hence is finitely generated. Therefore, $\mathrm{H}^*(A,k)$ is finitely generated.

□

Thus, this leads us to the question whether $\mathrm{H}^*(A,M)$ is a finitely generated module over $\mathrm{H}^*(A,k)$ where M is a finitely generated A-module? This is true in special cases for example, 1) A is a finite dimensional braided Hopf algebra [15], 2) A is the restricted enveloping algebra of a restricted Lie superalgebra [1], 3) A is a Frobenius-Lusztig kernel [5] and 4) A is the restricted enveloping algebra of a classical Lie superalgebra [14].

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