Oriented Hypergraphic Matrix-tree Type Theorems and Bidirected Minors via Boolean Order Ideals.

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Abstract

Given an integer matrix there is a natural oriented hypergraph associated to it. We examine restrictions of incidence-preserving maps to produce Sachs-Chaiken type All Minors Matrix-tree Theorems for oriented hypergraphic Laplacian and adjacency matrices. When the incidence structure is restricted to bidirected graphs the minor calculations are shown to correspond to principal order ideals of signed boolean lattices, and classical results are generalized or reclaimed.

Keywords: Matrix-tree theorem, Laplacian matrix, signed graph, bidirected graph, oriented hypergraph

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1. Introduction

An oriented hypergraph is a signed incidence structure that first appeared in [12] to study applications to VLSI via minimization and logic synthesis [11]. This incidence based approach allows for graph-theoretic theorems to be extended to hypergraphs using their locally signed graphic structure (see [8, 10, 9, 4]). An oriented hypergraph in which every edge has exactly 2 incidences is a *bidirected graph*, and can be regarded as orientations of signed graphs (see [6, 14, 15]).

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Recently, in [3], this incidence based approach was used to provide an integer-matrix version of Sachs' Coefficient Theorem via sub-incidence structures of oriented hypergraphs determined by the image of a class of maps called *contributors*. Moreover, the approach is universal to both the permanent and determinant of both the adjacency and Laplacian matrices by adjusting the incidence-injectivity of contributor maps.

Building on the work in [3], oriented hypergraphic Matrix-tree-type theorems are presented in Section 2.2. These theorems are incidence-hypergraphic variants of Sachs' Theorem [5, 1] and Chaiken's All Minors Matrix-tree Theorem [2]. Contributor maps are specialized to Laplacians of bidirected graphs in Section 3. A natural partial ordering of contributors is introduced where each associated equivalence classes (called *activation classes*) is boolean; this is done by introducing *incidence packing* and *unpacking* operations. If an oriented hypergraph contains edges of size larger than 3, then unpacking is not well-defined, and the resulting equivalence classes need not be lattices. Activation classes are further refined via iterated principal order ideals/filters in order to examine minors of the Laplacian.

Section 4 examines the contributors in the adjacency completion of a bidirected graph to obtain a restatement of the All Minors Matrix-tree Theorem in terms of sub-contributors (as opposed to restricted contributors). This implies there is a universal collection of contributors (up to resigning) which determines the minors of all bidirected graphs that have the same injective envelope — see [7] for more on the injective envelope. These sub-contributors determine permanents/determinants of the minors of the original bidirected graph and are activation equivalent to the forest-like objects in [2]. Additionally, the standard determinant of the signed graphic Laplacian is presented as a sum of maximal contributors, while the first minors of the Laplacian contain a subset of contributors that are activation equivalent to spanning trees.

While the techniques introduced for bidirected graphs do not readily extend to all oriented hypergraphs, they bear a remarkable similarity to Tutte's development of transpedances in [13] — indicating the possibility of a locally signed graphic interpretation of transpedance theory. Since contributor sets produce the finest possible set of objects signed $\{0, \pm 1\}$ whose sum produces the permanent/determinant it is natural to ask what classes of graphs achieve specific permanent/determinant values — see [3] for some elementary results on bounding by contributors.

2. Preliminaries and the Matrix-tree Theorem

2.1. Oriented Hypergraph Basics

A condensed collection of definitions are provided in this subsection to improve readability, for a detailed introduction to the definitions the reader is referred to [3]. An *oriented hypergraph* is a quintuple (V, E, I, ι, σ) where V, E, and I are disjoint sets of *vertices*, *edges*, and *incidences*, with *incidence function* $\iota : I \to V \times E$, and *incidence orientation function* $\sigma : I \to \{+1, -1\}$. A *bidirected graph* is an oriented hypergraph with the property that for each $e \in E$, $|\{i \in I \mid (proj_E \circ \iota)(i) = e\}| = 2$, and can be regarded as an orientation of a signed graph (see [14, 15]) where the *sign of an edge e* is

$$sgn(e) = -\sigma(i)\sigma(j),$$

where i and j are the incidences containing e.

A *backstep of G* is an embedding of $\overrightarrow{P}_1$ into G that is neither incidence-monic nor vertex-monic; a *loop of G* is an embedding of $\overrightarrow{P}_1$ into G that is incidence-monic but not vertex-monic; a *directed adjacency of G* is an embedding of $\overrightarrow{P}_1$ into G that is incidence-monic. A *directed weak walk of length k in G* is the image of an incidence preserving embedding of a directed path of length k into G. Conventionally, a backstep is a sequence of the form (v, i, e, i, v), a loop is a sequence of the form (v, i, e, j, v), and an adjacency is a sequence of the form (v, i, e, j, w) where $\iota(i) = (v, e)$ and $\iota(j) = (w, e)$. The *opposite* embedding is image of the reversal of the initial directed path, while the non-directed version is the set on the sequence's image.

The *sign of a weak walk* is defined as

$$sgn(W) = (-1)^k \prod_{h=1}^{2k} \sigma(i_h),$$

which implies that for a path in G the product of the adjacency signs of the path is equal to the sign of the path calculated as a weak walk.

2.2. The Matrix-tree Theorem

It was shown in [9] that the (v,w)-entry of the oriented incidence Laplacian are the negative weak walks of length 1 from v to w minus the number of positive weak walks of length 1 from v to w. This was restated in [3] as the follows:

Theorem 2.1. *The (v,w)-entry of $\mathbf{L}_G$ is $\sum_{\omega \in \Omega_1} -sgn(\omega(\vec{P}_1))$, where Ω_1 is the set of all incidence preserving maps $\omega : \vec{P}_1 \to G$ with $\omega(t) = v$ and $\omega(h) = w$.*

A *contributor of* G is an incidence preserving function $c : \coprod_{v \in V} \vec{P}_1 \to G$ with $p(t_v) = v$ and $\{p(h_v) \mid v \in V\} = V$. A *strong-contributor of* G is an incidence-monic contributor of G – i.e. the backstep-free contributors of G. Let $\mathfrak{C}(G)$ (resp. $\mathfrak{S}(G)$) denote the sets of contributors (resp. strong contributors) of G. In [3] contributors provided a finest count to determine the permanent/determinant of Laplacian and adjacency matrices and their characteristic polynomials of any integral matrix as the incidence matrix of the associated oriented hypergraph. The values $ec(c)$, $oc(c)$, $pc(c)$, and $nc(c)$ denote the number of even, odd, positive, and negative circles in the image of contributor c. Additionally, the sets $\mathfrak{C}_{\geq 0}(G)$ (resp. $\mathfrak{C}_{=0}(G)$) denote the set of all contributors with at least 0 (resp. exactly 0) backsteps.

Theorem 2.2 ([3]). *Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then*

1. $\text{perm}(\mathbf{L}_G) = \sum_{c \in \mathfrak{C}_{\geq 0}(G)} (-1)^{oc(c)+nc(c)}$,
2. $\det(\mathbf{L}_G) = \sum_{c \in \mathfrak{C}_{\geq 0}(G)} (-1)^{pc(c)}$,
3. $\text{perm}(\mathbf{A}_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{nc(c)}$,

4. $\det(\mathbf{A}_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{ec(c)+nc(c)}$.

For a $V \times V$ matrix $\mathbf{M}$, let $U, W \subseteq V$, define $[\mathbf{M}]_{(U;W)}$ be the minor obtained by striking out rows U and columns W from $\mathbf{M}$. Let $\mathfrak{C}(U;W;G)$ be the set of all sub-contributors of G with $c : \coprod_{u \in \overline{U}} \overrightarrow{P}_1 \to G$ with $p(t_u) = u$ and $\{p(h_u) \mid u \in \overline{U}\} = \overline{W}$. Define $\mathfrak{S}(U;W;G)$ analogously for strong-contributors. Let the values $en(c)$, $on(c)$, $pn(c)$, and $nn(c)$ denote the number of even, odd, positive, and negative, non-adjacency-trivial components (paths or circles) in the image of c.

Theorem 2.3. *Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then*

1. $\text{perm}([\mathbf{L}_G]_{(U;W)}) = \sum_{c \in \mathfrak{C}(U;W;G)} (-1)^{on(c)+nn(c)}$,
2. $\det([\mathbf{L}_G]_{(U;W)}) = \sum_{c \in \mathfrak{C}(U;W;G)} \varepsilon(c) \cdot (-1)^{on(c)+nn(c)}$,
3. $\text{perm}([\mathbf{A}_G]_{(U;W)}) = \sum_{c \in \mathfrak{S}(U;W;G)} (-1)^{nn(c)}$,
4. $\det([\mathbf{A}_G]_{(U;W)}) = \sum_{c \in \mathfrak{S}(U;W;G)} \varepsilon(c) \cdot (-1)^{en(c)+nn(c)}$.

Where $\varepsilon(c)$ is the number of inversions in the natural bijection from $\overline{U}$ to $\overline{W}$.

The proof of Theorem 2.3 is analogous to Theorem 4.1.1 in [3] using the bijective definitions of permanent/determinant.

The value of $\varepsilon(c)$ can be modified to count circle and paths separately, as the circle components simplify identical to the works in [3], thus part (2) of Theorem 2.3 can be restated as follows:

Theorem 2.4. *Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then*

$$\det([\mathbf{L}_G]_{(U;W)}) = \sum_{c \in \mathfrak{C}(U;W;G)} \varepsilon'(c) \cdot (-1)^{pc(c)} \cdot (-1)^{op(c)+np(c)}.$$

Where $\varepsilon'(c)$ is the number of inversions in paths parts of the natural bijection from $\overline{U}$ to $\overline{W}$.

Comparing, the non-zero elements of $\mathfrak{C}(U;W;G)$ are the Chaiken-type structures in [2] with multiplicities replaced with backstep maps.

3. Contributor Structure of Bidirected Graphs

3.1. Pre-contributors and Incidence Packing

Throughout this section, G is a bidirected graph in which every connected component contains at least one adjacency, and $\overrightarrow{P}_1$ is a directed path of length 1. A *pre-contributor of* G is an incidence preserving function $p : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ with $p(t_v) = v$. That is, the disjoint union of $|V|$ copies of $\overrightarrow{P}_1$ into G such that every tail-vertex labeled by v is mapped to v.

Consider a pre-contributor p with $p(t_v) \neq p(h_v)$ for vertex $v \in V$. *Packing a directed adjacency of a pre-contributor p into a backstep at vertex v* is a pre-contributor p_v such that $p_v = p$ for all $u \in V \smallsetminus v$, and for vertex v

$$\begin{aligned} p((\overrightarrow{P}_1)_v) &= (v, i, e, j, w),\ i \neq j, \\ \text{and } p_v((\overrightarrow{P}_1)_v) &= (v, i, e, i, v). \end{aligned}$$

Thus, the head-incidence and head-vertex of adjacency $p((\overrightarrow{P}_1)_v)$ are identified to the tail-incidence and tail-vertex.

Unpacking a backstep of a pre-contributor p into an adjacency out of vertex v is a pre-contributor p^v is defined analogously but for vertex v, the head-incidence and head-vertex of backstep $p((\overrightarrow{P}_1)_v)$ are identified to the unique incidence and vertex that would complete the adjacency in bidirected graph G. Note that this is unique for a bidirected graph since every edge has exactly two incidences, but this is not the case in if there are edges of size greater than 2.

For a bidirected graph G and vertex v, let $\mathfrak{P}(G)$ be the set of all pre-contributors of G, $\mathfrak{P}_v(G)$ be the set of pre-contributors with a backstep at v, and let $\mathfrak{P}^v(G)$ be the set of pre-contributors with a directed adjacency from v.

Lemma 3.1. *Packing and unpacking are inverses between $\mathfrak{P}_v$ and $\mathfrak{P}^v$.*

PROOF. By definition $(p_v)^v = p$ and $(p^v)_v = p$ for appropriate contributors in $\mathfrak{P}^v$ or $\mathfrak{P}_v$. □

Lemma 3.2. *Packing is commutative.*

PROOF. Let $p \in \mathfrak{P}^v \cap \mathfrak{P}^w$, $p_{vw} := p_w \circ p_v$, and $p_{wv} := p_v \circ p_w$. By definition, $p_{vw} = p_{wv}$ for all $(\overrightarrow{P}_1)_u$ with $u \in V \setminus \{v, w\}$. For vertices v and w,

$$\begin{aligned} p_v((\overrightarrow{P}_1)_w) &= p((\overrightarrow{P}_1)_w), \\ \text{and } p_w((\overrightarrow{P}_1)_v) &= p((\overrightarrow{P}_1)_v). \end{aligned}$$

Giving,

$$\begin{aligned} p_{vw}((\overrightarrow{P}_1)_w) &= p_w((\overrightarrow{P}_1)_w) = p_{wv}((\overrightarrow{P}_1)_w), \\ \text{and } p_{vw}((\overrightarrow{P}_1)_v) &= p_v((\overrightarrow{P}_1)_v) = p_{wv}((\overrightarrow{P}_1)_v). \end{aligned}$$

□

Lemma 3.3. *Unpacking is commutative.*

PROOF. Proof is identical to packing after reversing subscript and superscripts. □

3.2. Contributors and Activation

A contributor of G is a pre-contributor where $\{p(h_v) \mid v \in V\} = V$.

For each $c \in \mathfrak{C}(G)$ let $tc(c)$ be the total number of circles in c; a degenerate 2-circle (a closed 2-weak-walk) is considered a circle, while a degenerate 1-circle (a backstep) is not. *Activating a circle of contributor c* is a minimal sequence of unpackings that results in a new contributor c' such that $tc(c) = tc(c') - 1$. *Deactivating a circle of contributor c* is a minimal sequence of packings that results in a new contributor c'' such that $tc(c) = tc(c'') + 1$. Immediately from the definition we have:

Lemma 3.4. *Let $c, d \in \mathfrak{C}(G)$. Contributor d can be obtained by activating circles of c if, and only if, c can be obtained by deactivating a circles of d. Moreover, the activation/deactivation sets are equal.*

Define the *activation partial order* $\leq_a$ where $c \leq_a d$ if d is obtained by a sequence of activations starting with c, and the *activation equivalence relation* $\sim_a$where $c \sim_a d$ if $c \leq_a d$ or $d \leq_a c$. The elements of $\mathfrak{C}(G)/\sim_a$ are called the *activation classes of G*.

Example 1. *Figure 1 is a bidirected graph (with incidences omitted), and some contributors are depicted, sorted by their associated permutation.*

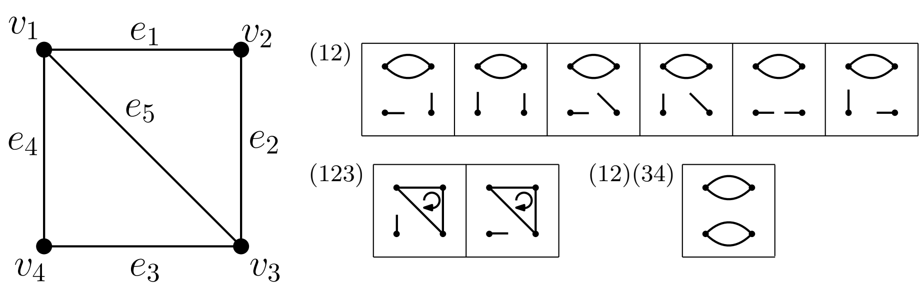

Figure 1: Understanding contributors.

Observe that the fifth contributor in (12) *is below the* (12)(34) *contributor in the activation partial order.*

Lemma 3.5. *The minimal elements of activation classes are incomparable, consist of only backsteps, and correspond to the identity permutation.*

Lemma 3.6. *All activation classes of G are boolean lattices.*

Proof. For a given activation class consider the set of possible active circles. The elements of each activation class is ordered by the subsets of active circles, with unique maximal element having all circles active, and unique minimal element having all circle inactive. □

We have the following lemma using the facts that (1) every connected component of G is assumed to have an adjacency, and (2) every contributor corresponds to a permutation on the vertices, there is at least one circle that can be activated for each minimal element in each activation class.

Lemma 3.7. *Each maximal contributor in a activation class contains at least* 1 *circle.*

Corollary 3.8. *Each activation class has at least* 2 *members.*

Example 2. *Figure 2 shows* 3 *activation classes of the graph from Example 1.*

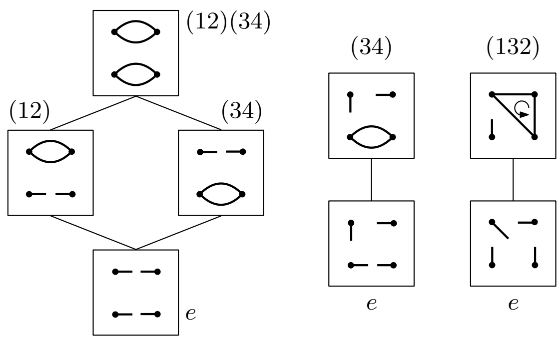

Figure 2: Activation classes are boolean.

The activation classes are ranked by the number of circles, and the minimal element corresponds to the identity permutation.

3.3. Partitioning Activation Classes

For $u, w \in V$ two contributors c and d are *uw-equivalent*, denoted $c \sim_{uw} d$, if $c(h_u) = d(h_u) = w$. Since $\sim_{uw}$ only collects contributors in which the image of $(\overrightarrow{P}_1)_u$ has head-vertex mapped to w we have:

Lemma 3.9. $\mathfrak{C}(G)/(\sim_{u_1w_1} \circ \sim_{u_2w_2}) = \mathfrak{C}(G)/(\sim_{u_2w_2} \circ \sim_{u_1w_1})$ *for* $u_1, w_1, u_2, w_2 \in V$.

As w varies, the composition $\sim_{u\bullet} := \bigcirc_{w \in V} \sim_{uw}$ is well defined without the need for a total ordering on V. Moreover,

Lemma 3.10. $\mathfrak{C}(G)/(\sim_{uw} \circ \sim_a) = \mathfrak{C}(G)/(\sim_a \circ \sim_{uw})$ *or any* $u, w \in V$.

By construction, the relation $\sim_{uw} \circ \sim_a$ (subsequently, $\sim_{u\bullet} \circ \sim_a$) refines each $\mathcal{A} \in \mathfrak{C}(G)/\sim_a$. Let $\mathcal{A}/\sim_{uw}$ and $\mathcal{A}/\sim_{u\bullet}$ denote the refinement of $\mathcal{A}$ by $\sim_{uw}$ or $\sim_{u\bullet}$ respectively.

Theorem 3.11. *$\mathfrak{C}(G)/(\sim_{u\bullet} \circ \sim_a)$ is a refinement of each activation class in $\mathfrak{C}(G)/\sim_a$ into two principal order ideals (one upper and one lower, with the upper order ideal possibly empty) that are boolean complements. Moreover, the upper order ideal of activation class $\mathcal{A}$ is empty if, and only if, $c(h_u) = u$ for all $c \in \mathcal{A}$.*

Proof. Let $\mathcal{A} \in \mathfrak{C}(G)/\sim_a$ and observe that every least element of $\mathcal{A}$ is an adjacency free contributor, so the set of possible maximal elements such that $c(h_u) = u$ is non-empty. Using the definition of activation, the facts that $\mathcal{A}$ is boolean, and that there is at least one element (the **0**-element) in each activation class with $h_u \to u$, there is exactly one maximal element with $h_u \to u$, let $M(u;u;\mathcal{A})$ be this maximal element. Thus, the principal ideal $\downarrow M(u;u;\mathcal{A})$ exists and is necessarily boolean. Moreover, $\downarrow M(u;u;\mathcal{A}) = \mathcal{A}$ if, and only if, $c(h_u) = u$ for all contributors $c \in \mathcal{A}$, thus $\mathcal{A}/\sim_{uw}$ is empty for all $w \neq u$.

Since $\mathcal{A}$ is boolean, if there is a contributor d such that $d(h_u) = w \neq u$, then all contributors of $\mathcal{A}$ with $h_u \nrightarrow u$ must have $h_u \to w$, since every edge is a 2-edge. Moreover, if there is a contributor of $\mathcal{A}$ with $h_u \to w$, then there is a unique minimal element with $m(h_u) = w \neq u$, let $m(u;w;\mathcal{A})$ be this minimal element (if it exists). By construction, $m(u;w;\mathcal{A})$ is the contributor of $\mathcal{A}$ with only the circle containing the uw-adjacency active, is a rank 1 element in $\mathcal{A}$, and is the boolean complement of $M(u;u;\mathcal{A})$. Thus, $\mathcal{A} = \downarrow M(u;u;\mathcal{A}) \cup \uparrow m(u;w;\mathcal{A})$. □

The $(u;w)$-*cut of activation class* $\mathcal{A}$ is the subclass of $\mathcal{A}/\sim_{uw}$ where each element has $c(h_u) = w$ — that is, $\downarrow M(u;u;\mathcal{A})$ if $u = w$, or $\uparrow m(u;w;\mathcal{A})$ if $u \neq w$ and $m(u;w;\mathcal{A})$ exists. Let $U, W \subseteq V$ with $|U| = |W| = k$, and $\mathbf{u} = (u_1, u_2, \ldots, u_k)$, $\mathbf{w} = (w_1, w_2, \ldots, w_k)$ be linear orderings of U and W according to their placement in the implied linear ordering of V in the underlying incidence matrix. The $(\mathbf{u};\mathbf{w})$-*cut of the activation class* $\mathcal{A}$ is the corresponding subclass in $\mathcal{A}/\bigcirc_{i\in[k]} \sim_{u_i w_i}$. Let $\mathcal{A}(\mathbf{u};\mathbf{w};G)$ denote the $(\mathbf{u};\mathbf{w})$-cut of activation class $\mathcal{A}$, and $\widehat{\mathcal{A}}(\mathbf{u};\mathbf{w};G)$ be the elements of $\mathcal{A}(\mathbf{u};\mathbf{w};G)$ with the adjacency or backstep from u_i to w_i is removed for each i. Let $\mathfrak{C}(\mathbf{u};\mathbf{w};G)$ be the set of all elements in all $\mathcal{A}(\mathbf{u};\mathbf{w};G)$, and $\widehat{\mathfrak{C}}(\mathbf{u};\mathbf{w};G)$ be the elements of $\mathfrak{C}(\mathbf{u};\mathbf{w};G)$ with the adjacency or backstep from u_i to w_i is removed for each i.

Example 3. *Figure 3 shows (v_1, v_1)-cuts of the contribution classes from Figure 2.*

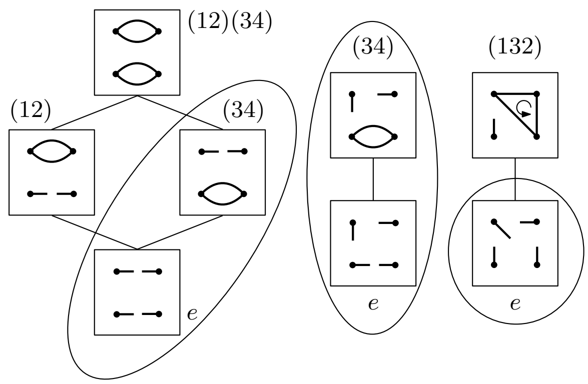

Figure 3: (v_1, v_1)-cuts of contribution classes.

Observe that the first two sub-classes are non-trivial boolean lattices, the final sub-class is a trivial boolean lattices, and the second sub-class has an empty upper order ideal.

4. Universal Contributors and The Matrix-tree Theorem

4.1. Adjacency Completion

Given an oriented hypergraph $G = (V, E, I, \iota, \sigma)$ let $G' = (V, E \cup E_0, I \cup I_0, \iota, \sigma')$ be the oriented hypergraph obtained by adding a bidirected edge to G for every non-adjacent pair of vertices, where $\sigma' = \sigma$ for all $i \in I$, and $\sigma' = 0$ for all $i \in I_0$ (see [7] for relationship to the injective envelope). The *sign of a (sub-)contributor* is the product of the weak walks. The inclusion of 0-signed-incidences in G' implies that an element of $\widehat{\mathfrak{C}}(\mathbf{u}; \mathbf{w}; G)$ has non-zero sign if, and only if, it exists in G. Let $\widehat{\mathfrak{C}}_{\neq 0}(\mathbf{u}; \mathbf{w}; G')$ be the set of non-zero elements of $\widehat{\mathfrak{C}}(\mathbf{u}; \mathbf{w}; G')$. This fact gives the following simple Lemma that relates the global contributors of G' to the Chaiken-type forests of [2] separated by multiplicity:

Lemma 4.1. *If $U, W \subseteq V$ with with linear orderings* $\mathbf{u}$ *and* $\mathbf{w}$*, then*

$$\widehat{\mathfrak{C}}_{\neq 0}(\mathbf{u}; \mathbf{w}; G') = \mathfrak{C}(U; W; G).$$

Example 4. *Figure 4 shows an additional contribution class from Figure 2 that exists in the adjacency completion.*

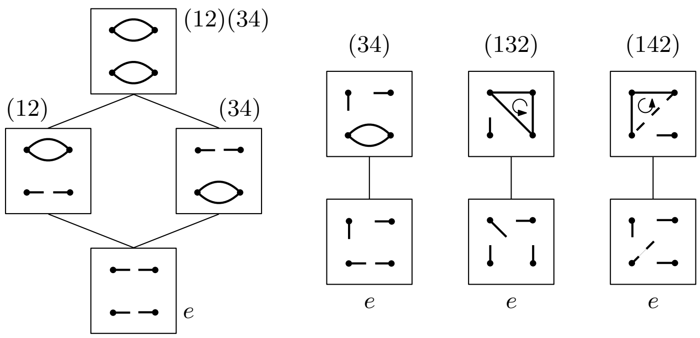

Figure 4: Contributors in G'.

Observe that the (142) *contributor with the* v_4v_2*-adjacency removed is a member of* $\widehat{\mathfrak{C}}_{\neq 0}(v_4; v_2; G')$*, exists in* G*, and counts in the* v_4v_2*-minor calculation.*

Lemma 4.1 provides the following restatement of Theorem 2.3:

Theorem 4.2. *Let G a bidirected graph with Laplacian matrix* $\mathbf{L}_G$*. Given* $U, W \subseteq V$ *with* $|U| = |W|$ *and linear orderings* $\mathbf{u}$ *and* $\mathbf{w}$*, let* $[\mathbf{L}_G]_{(\mathbf{u};\mathbf{w})}$ *be the minor of* $\mathbf{L}_G$ *formed by the ordered deletion of the rows corresponding to the vertices in* U *and the columns corresponding to the vertices in* W*. Then we have,*

1. $perm([\mathbf{L}_G]_{(\mathbf{u};\mathbf{w})}) = \sum_{c \in \widehat{\mathfrak{C}}_{\neq 0}(\mathbf{u};\mathbf{w};G')} (-1)^{on(c)+nn(c)}$,
2. $\det([\mathbf{L}_G]_{(\mathbf{u};\mathbf{w})}) = \sum_{c \in \widehat{\mathfrak{C}}_{\neq 0}(\mathbf{u};\mathbf{w};G')} \varepsilon(c) \cdot (-1)^{on(c)+nn(c)}$.

Where $\varepsilon(c)$ *is the number of inversions in the natural bijection from* $\overline{U}$ *to* $\overline{W}$*.*

4.2. Applications

We now examine alternate proofs of established results using the boolean order of contributor maps. Besides providing more insight into contributors, the hope is these techniques can be generalized to a complete theory for oriented hypergraphs — as evidenced by Theorem 2.3.

Let $\mathfrak{M}^-$ be the set of maximal elements from each activation class that contains a negative circle.

Lemma 4.3. *If G is a signed graph, then* $\det(\mathbf{L}_G) = \sum_{c \in \mathfrak{M}^-} 2^{nc(c)}$.

Proof. From Theorem 2.2 and Lemma 3.6

$$\begin{aligned} \det(\mathbf{L}_G) &= \sum_{c \in \mathfrak{C}_{\geq 0}(G)} (-1)^{pc(c)} \\ &= \sum_{\mathcal{A} \in \mathfrak{C}(G)/\sim_a} \sum_{c \in \mathcal{A}} (-1)^{pc(c)}. \end{aligned}$$

While Corollary 3.8 implies that every activation class has at least 2 elements, so every activation class has a covering by 2-chains. Let $M_{\mathcal{A}}^-$ be the element of activation class $\mathcal{A}$ that has the maximal number of negative circles. Since $\mathcal{A}$ is boolean, $M_{\mathcal{A}}^-$ is unique (if it exists), and $\downarrow M_{\mathcal{A}}^-$ is also boolean. Take any covering of $\mathcal{A}$ by 2-chains that also contains a covering of $\downarrow M_{\mathcal{A}}^-$ by 2-chains. Every 2-chain outside of $\downarrow M_{\mathcal{A}}^-$ sums to zero, while there are exactly $nc(M_{\mathcal{A}}^-)$ 2-chains remaining in $\downarrow M_{\mathcal{A}}^-$ that do not cancel. The result follows with the observation that $M_{\mathcal{A}}^-$ and the maximal element of $\mathcal{A}$ have the same number of negative circles. □

Corollary 4.4. *If G is a balanced signed graph, then* $\det(\mathbf{L}_G) = 0$.

Let $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ be the set of non-zero elements of $\widehat{\mathcal{A}}(u; w; G')$.

Lemma 4.5. *If G is a bidirected graph, then the set of elements in all single-element $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ is activation equivalent to the set of spanning trees of G.*

Proof. *Case 1a ($u = w$):* Suppose $u = w$, and let $\widehat{\mathcal{A}}_{\neq 0}(u; u; G')$ be a single-element activation classes of G'. The element of $\widehat{\mathcal{A}}_{\neq 0}(u; u; G')$ consists of exactly $|V|-1$ backsteps, all of which exist in G, but none of which contain u. Unpacking all backsteps results in a circle-free subgraph on $|V|$ vertices with $|V| - 1$ edges, i.e. a spanning tree — if it was not circle-free then $\widehat{\mathcal{A}}_{\neq 0}(u; u; G')$ would have more than a single element. Thus, the cardinality of the set of single-element $\widehat{\mathcal{A}}_{\neq 0}(u; u; G')$ is less than, or equal to, the number spanning trees of G.

Case 1b ($u = w$): Now consider all spanning outward arborescences of G rooted at u. For each arborescence, pack all adjacencies along the opposite orientation of the arborescence to produce a unique, non-zero, element of an $\widehat{\mathcal{A}}(u; u; G')$. Thus, the cardinality of the set of single-element $\widehat{\mathcal{A}}_{\neq 0}(u; u; G')$ is greater than, or equal to, the number spanning trees of G.

Case 2a ($u \neq w$): Suppose $u \neq w$, and let $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ be a single-element activation class of G'. Since $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ is obtained by the upper order ideal of $\mathcal{A}(u; w; G')$ generated by the maximal contributor $M(u; w; \mathcal{A})$, and $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ consists of a single element, $M(u; w; \mathcal{A})$ must be unicyclic. Thus, the corresponding contributor in $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ must contain a wu-path on k vertices, and backsteps at the $|V| - k$ vertices outside the path. Unpacking all backsteps is a circle-free subgraph on $|V|$ vertices and $|V|-1$ edges — if not then $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ would have more than a single element. Thus, the cardinality of the set of single-element $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ is less than, or equal to, the number spanning trees of G.

Case 2b ($u \neq w$): Now consider the set of spanning trees of G. To see that the cardinality of the set of single-element $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ is greater than, or equal to, the number spanning trees of G, examine the following sub-cases:

Case 2b, part 1 ($u \neq w$): If a spanning tree T contains an adjacency between u and w construct the unicyclic-contributor of G as follows: (1) add another parallel adjacency between u and w, (2) orient the parallel edges to form a degenerate 2-circle, and (3) pack all remaining adjacencies away from u and w. The member of $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ is obtained by deleting the uw-directed-adjacency.

Case 2b, part 2 ($u \neq w$): If a spanning tree T does not contain an adjacency between u and w construct the unicyclic-contributor of G as follows: (1) add a uw-directed-adjacency, (2) oriented the resulting unique fundamental circle coherently with the uw-directed-adjacency, and (3) pack all remaining adjacencies away from the fundamental circle. The member of $\widehat{\mathcal{A}}_{\neq 0}(u;w;G')$ is obtained by deleting the uw-directed-adjacency. ◻

Using the techniques of the previous two Lemmas, adjusting for the cofactor, and using the fact that every adjacency (hence, circle) is positive in a graph, it is easy to reclaim:

Corollary 4.6. *If G is a graph, then the uw-cofactor of $\mathbf{L}_G$ is $T(G)$, the number of spanning trees of G.*

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